Closed Form Solution for SHM-Based Bayesian Reliability Assessment

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ABSTRACT

Extensive structural health monitoring (SHM) of civil infrastructure provides massive amounts of data that must be promptly and effectively analyzed through appropriate models to infer the health state of structures. In general terms, the health state of a structure can be expressed as its structural reliability with respect to the most significant limit states. When the reliability is estimated based on SHM data, the problem is typically solved numerically with iterative approaches, which are computationally expensive and do not allow early warning and prompt response. This work presents a logically consistent approach for the Bayesian estimation of structural reliability based on SHM observations. Moreover, it illustrates how closed-form solutions can be obtained using linear models and Normal random variables. The proposed approach can effectively evaluate the sensitivity of structural reliability with respect to SHM observations; this can be used as a novel performance index for monitoring systems. Finally, this paper proposes an application of this approach to a real-life case study, the crack opening monitoring of the Settefonti highway viaduct in Italy.

INTRODUCTION

There is a general agreement among researchers and infrastructure managers regarding the role of Structural Health Monitoring (SHM) in providing helpful information on the Structural Reliability of infrastructures. SHM aims to extract the damaged state of infrastructures from observations exploiting structural models. On the other hand, Structural Reliability aims to estimate the probability of failure of a given Limit State, i.e., a requirement a structure should satisfy during its service life.

Demand from infrastructure managers for monitoring systems on structures, especially bridges, has increased over the last decades. Indeed, regulators are interested in improving the structural health state of bridges and upgrading inspection and monitoring procedures. Therefore, a scalable SHM framework is needed to handle the increasing amount of data and to provide timely the structural reliability.

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To assess the reliability levels of infrastructures, it is necessary to obtain information about the structural health state based on SHM Observations using appropriate models. Bayesian updating of State Parameters based on SHM Observations is widely discussed in the literature [1], [2]. To better handle the probabilistic dependencies among the variables involved in the Bayesian frameworks, it is helpful to formulate the problem in terms of Bayesian Networks (BNs) [3]. BNs are a Probabilistic Graphical Model type that represents variables and their mutual correlations through a joint probability distribution. In the SHM field, relevant works regarding using BNs are [4], [5].

To summarize, several frameworks have been proposed for the Bayesian updating of structural reliability based on SHM data. Most of these use numerical solutions such as Monte Carlo simulation, MCMC, or Subset simulations. Other works use Gaussian Bayesian Networks (GBNs) [6], [7], which can offer direct solutions; however, GBNs can lead to errors due to the strong assumptions about the distribution of variables and the linearity of models.

As mentioned, managers need prompt and scalable methods to assess infrastructure reliability based on SHM data. Existing numerical methods can be computationally inefficient; therefore, closed-form formulations should be reconsidered. This paper proposes a framework based on BNs for structural reliability analysis using SHM data. This framework provides a closed-form solution for structural reliability estimation based on the linearity of models and the Gaussian distribution of random variables. Finally, we propose an application of the framework to a real-life case study, the Settefonti highway viaduct in northern Italy. The application focuses on the crack opening limit state of the concrete.

PROBLEM STATEMENT AND FRAMEWORK DEFINITION

When a manager decides to install a SHM system on a structure, it aims to update the knowledge of its structural reliability. A SHM system is a network of sensors installed on the structure, which collects information about the configuration of the structure. Each sensor provides measurements of the observed physical quantities (e.g., displacements, rotations, temperatures). These measurements are adequately collected in the Observation vector $\boldsymbol{y}$. From the Observations, the manager can infer the structural health state of the structure, represented by specific State Parameters (e.g., section stiffness, corrosion level of steel reinforcement); these are collected into the State Parameters vector $\boldsymbol{\theta}$. Interpretative model $\boldsymbol{y} = \mathcal{M}(\boldsymbol{\theta})$ represents the relationship between the State Parameters and the Observations.

The reliability assessment of the structure with respect to the desired Limit State involves specific parameters, which are collected in the Limit State Parameters vector $\boldsymbol{x}$. The Limit State Parameters are related to the State Parameters $\boldsymbol{\theta}$ through the Analysis Model $\boldsymbol{x} = \mathcal{A}(\boldsymbol{\theta})$. The Limit State Parameters are involved in the Limit State Function $z = \mathcal{Z}(\boldsymbol{x})$, which mathematically describes the desired Limit State.

The Probability of Failure p_f of a Limit State coincides with the probability that, given the probability distribution of $\boldsymbol{x}$, the realization of the Limit State Function z takes on a negative value. The failure probability p_f is commonly evaluated by exploiting Second Moment Reliability Methods [8], where p_f is defined in function of the Reliability Index β, i.e., the ratio between the mean μ_z and the standard deviation σ_z of z.

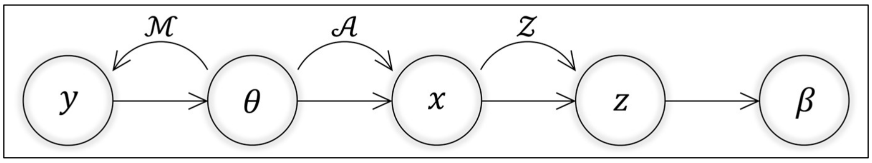

Figure 1. Framework for strutural reliability assessment based on SHM data.

As the reader may know, given two systems with different Reliability Index β, the most reliable system is the one that holds the greatest value of β.

By sequentially applying the models defined above, it is possible to express the Reliability Index β as a function of the Observations vector y. This process is highlighted in Figure 1, which shows this process from left to right.

Let us consider $\boldsymbol{\theta}$, $\boldsymbol{y}$, and $\boldsymbol{x}$ as random variables (RVs); consequently, their relationship (represented in Figure 1) can be modeled as a directed, a-cyclic BN. In the following Sections, we present the framework formulation under the assumptions of linear models and Gaussian RVs. Then, we propose the framework application to the case of the Limit State of cracking in prestressed concrete beams.

FORMULATION

The framework defined above is based on the following assumptions: RVs are continuous (Hp1); RVs are Normally distributed (Hp2); RVs are interconnected by linear models (Hp3). Based on Hp1, the Observation $\boldsymbol{y}$, the State Parameters $\boldsymbol{\theta}$, and the Limit State Parameters $\boldsymbol{x}$ are defined as vectors with real components. In addition, based on Hp2, their follows a Normal distribution $\mathcal{N}(\,\cdot \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\,)$, where μ and $\boldsymbol{\Sigma}$ are the mean and the covariance matrix of the distribution, respectively. Depending on data availability, the variables $\boldsymbol{y}$, $\boldsymbol{\theta}$, and $\boldsymbol{x}$ follow their prior probability distribution – before measurements are available, or their posterior probability distribution – after measurements are available. Finally, based on Hp3, these variables are connected by linear models – a linear combination of input parameters.

Let us start with the State Parameters $\boldsymbol{\theta}$: its prior distribution is described by its mean $\boldsymbol{\mu}_\theta$ and covariance matrix $\boldsymbol{\Sigma}_\theta$ and do not depend on the structural configuration. Then, let us move on to the Observation $\boldsymbol{y}$ and define the Interpretative Model $\mathcal{M}(\boldsymbol{\theta})$ as:

$$\boldsymbol{y} = \boldsymbol{\mu}_y + \mathbf{D}\,(\boldsymbol{\theta} - \boldsymbol{\mu}_\theta) + \boldsymbol{e}_y \tag{1}$$

where $\mathbf{D}$ is the Sensitivity Matrix of the Interpretative Model, $\boldsymbol{e}_y$ is the random error of the Interpretative Model, and $\boldsymbol{\mu}_y$ and $\boldsymbol{\mu}_\theta$ are the mean of the prior distributions of $\boldsymbol{y}$ and $\boldsymbol{\theta}$, respectively. The error $\boldsymbol{e}_y$ includes model and measurement errors and is defined as a zero-mean multivariate Normal density function with covariance matrix $\boldsymbol{\Sigma}_y$.

The mean of the prior distributions of $\boldsymbol{y}$ reads as:

$$\boldsymbol{\mu}_y = \boldsymbol{y}_0 + \mathbf{D}\,\boldsymbol{\mu}_\theta \tag{2}$$

Let us move to the Limit State Parameters $\boldsymbol{x}$ and define the Analysis Model $\mathcal{A}(\boldsymbol{\theta})$ as:

$$\boldsymbol{x} = \boldsymbol{\mu}_x + \mathbf{A}\,(\boldsymbol{\theta} - \boldsymbol{\mu}_\theta) + \boldsymbol{e}_x \tag{3}$$

where, $\mathbf{A}$ is the Sensitivity Matrix of the Analysis Model and $\boldsymbol{e}_x$ is the random error of the Analysis Model, defined as a zero-mean multivariate Normal density function with a covariance matrix $\boldsymbol{\Sigma}_x \in \mathbb{R}^{K\times K}$. The mean $\boldsymbol{\mu}_x$ of the prior distribution of $\boldsymbol{x}$ reads as:

$$\boldsymbol{\mu}_x = \boldsymbol{x}_0 + \mathbf{A}\,\boldsymbol{\mu}_\theta \tag{4}$$

In light of this, the BN defined in the previous Section becomes a GBN [9]. The GBN expressing the relationship between $\boldsymbol{\theta}$, $\boldsymbol{y}$, and $\boldsymbol{x}$ to is shown in Figure 2. In this GBN, $\boldsymbol{y}$ is an observed variable, while $\boldsymbol{\theta}$ and $\boldsymbol{x}$ are not observed and must be estimated. Moving from $\boldsymbol{y}$ to $\boldsymbol{x}$, we perform (i) probabilistic inference from $\boldsymbol{y}$ to $\boldsymbol{\theta}$ to determine the posterior distribution of $\boldsymbol{\theta}$ given $\boldsymbol{y}$; (ii) uncertainty propagation from $\boldsymbol{\theta}$ to $\boldsymbol{x}$ to determine the posterior distribution of x given $\boldsymbol{y}$. To solve these steps, we must define the prior precision matrices $\boldsymbol{\Lambda}_\theta, \boldsymbol{\Lambda}_x$ and $\boldsymbol{\Lambda}_y$ – the inverse of the prior covariance matrices $\boldsymbol{\Sigma}_\theta$, $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_y$ – which quantifies the level of accuracy of the respective variable.

Let us start solving the BN by tackling the probabilistic inference step. Equation (5) shows the solution [9], [10], where $\boldsymbol{\mu}_{\theta|y}$ and $\boldsymbol{\Sigma}_{\theta|y}$ are the mean and the covariance matrix of the posterior distribution of $\boldsymbol{\theta}$ given $\boldsymbol{y}$, respectively.

$$\boldsymbol{\Sigma}_{\theta|y} = \left[\boldsymbol{\Lambda}_\theta + \boldsymbol{D}^T\,\boldsymbol{\Lambda}_y\,\boldsymbol{D}\right]^{-1};\;\; \boldsymbol{\mu}_{\theta|y} = \boldsymbol{\mu}_\theta + \boldsymbol{\Sigma}_{\theta|y}\,\boldsymbol{D}^T\,\boldsymbol{\Lambda}_y\,(\boldsymbol{y} - \boldsymbol{\mu}_y)\,; \tag{5}$$

Now, let us tackle the uncertainty propagation step. Equation (6) shows the solution based on the theory of error propagation [9]. Here, $\boldsymbol{\mu}_{x|y}$ and $\boldsymbol{\Sigma}_{x|y}$ are the mean and the covariance matrix of the posterior distribution of $\boldsymbol{x}$ given $\boldsymbol{y}$, respectively.

$$\begin{aligned} \boldsymbol{\Sigma}_{x|y} &= \boldsymbol{\Sigma}_x + \mathbf{A}\,\boldsymbol{\Sigma}_{\theta|y}\,\mathbf{A}^T\,; \\ \boldsymbol{\mu}_{x|y} &= \boldsymbol{x}_0 + \mathbf{A}\,\boldsymbol{\mu}_\theta + \mathbf{A}\,\boldsymbol{\Sigma}_{\theta|y}\,\mathbf{D}^T\,\boldsymbol{\Lambda}_y\,(\boldsymbol{y} - \boldsymbol{\mu}_y)\,; \end{aligned} \tag{6}$$

Regarding the reliability assessment, the Limit State Function $\mathcal{Z}$ depends only on the Limit State Parameters $\boldsymbol{x}$; consequently, the reliability of the structure depends only on the distribution of $\boldsymbol{x}$. Considering Hp3, $\mathcal{Z}$ is linear with respect to $\boldsymbol{x}$ and defined as in Equation (7), where $b_0 \in \mathbb{R}$ is the offset and $\boldsymbol{b} \in \mathbb{R}^K$ is the sensitivity parameter.

$$z = b_0 + \boldsymbol{b}^T\boldsymbol{x} \;\in\; \mathbb{R} \tag{7}$$

Since x is a random variable, z is a random variable too; and since $\boldsymbol{x}$ follows a Normal distribution and $\mathcal{Z}$ is linear, z follows a Normal distribution too, with mean $\boldsymbol{\mu}_z$ and variance σ_z^2. When $\boldsymbol{x}$ follows its posterior distribution, the mean and variance of z becomes $\mu_{z|y}$ and $\sigma_{z|y}^2$, respectively, and are defined as in Equation (8):

$$\begin{aligned} \mu_{z|y} &= b_0 + \boldsymbol{b}^T\boldsymbol{x}_0 + \boldsymbol{b}^T\mathbf{A}\,\boldsymbol{\mu}_\theta + \boldsymbol{b}^T\mathbf{A}\,\boldsymbol{\Sigma}_{\theta|y}\,\mathbf{D}^T\boldsymbol{\Lambda}_y\,(\boldsymbol{y} - \boldsymbol{\mu}_y)\,; \\ \sigma_{z|y}^2 &= \boldsymbol{b}^T\,\boldsymbol{\Sigma}_{x|y}\,\boldsymbol{b} = \boldsymbol{b}^T\,\boldsymbol{\Sigma}_x\,\boldsymbol{b} + \boldsymbol{b}^T\mathbf{A}\,\boldsymbol{\Sigma}_{\theta|y}\,\mathbf{A}^T\boldsymbol{b}\,; \end{aligned} \tag{8}$$

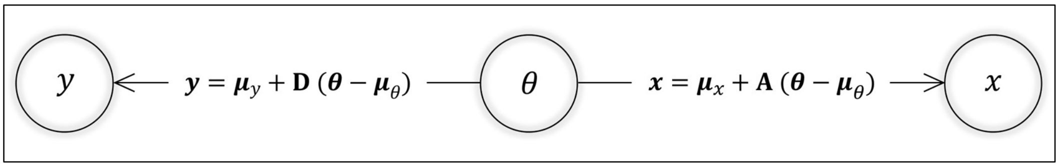

Figure 2. GBN expressing the relationship between $\boldsymbol{\theta}$, $\boldsymbol{y}$, and $\boldsymbol{x}$.

Finally, we can express the Reliability Index β as a function of $\boldsymbol{y}$ based on Equation (7). Their relation is linear and is reported in Equation (9), where β_0 is the offset of the Reliability Index β and $\boldsymbol{\nabla\beta}$ is the Reliability Index Sensitivity.

$$\beta = \beta_0 + \boldsymbol{\nabla\beta}\left(\boldsymbol{y} - \boldsymbol{\mu}_y\right) \tag{9}$$

In particular, β_0 is the value that β assumes when the Observation $\boldsymbol{y}$ is equal to $\boldsymbol{\mu}_y$, while $\boldsymbol{\nabla\beta}$ represents the variation of β given a unit variation of $\boldsymbol{y}$. Hence, the product between $\boldsymbol{\nabla\beta}$ and $\boldsymbol{y}$ returns the variation of the Reliability Index, $\Delta\beta$, given $\boldsymbol{y}$. Equation (10) defines $\beta_0 \in \mathbb{R}$ and $\boldsymbol{\nabla\beta} \in \mathbb{R}^M$.

$$\beta_0 = \frac{b_0 + \boldsymbol{b}^T\boldsymbol{x}_0 + \boldsymbol{b}^T\mathbf{A}\,\boldsymbol{\mu}_\theta}{\sqrt{\boldsymbol{b}^T\,\boldsymbol{\Sigma}_x\,\boldsymbol{b} + \boldsymbol{b}^T\mathbf{A}\,\boldsymbol{\Sigma}_{\theta|y}\,\mathbf{A}^T\boldsymbol{b}}}; \qquad \boldsymbol{\nabla\beta} = \frac{\boldsymbol{b}^T\mathbf{A}\,\boldsymbol{\Sigma}_{\theta|y}\,\mathbf{D}^T\boldsymbol{\Lambda}_y}{\sqrt{\boldsymbol{b}^T\,\boldsymbol{\Sigma}_x\,\boldsymbol{b} + \boldsymbol{b}^T\mathbf{A}\,\boldsymbol{\Sigma}_{\theta|y}\,\mathbf{A}^T\boldsymbol{b}}}; \tag{10}$$

APPLICATION AND DISCUSSION

In this Section, we discuss the application of the proposed framework to a real case study, the Settefonti highway viaduct in Italy. The Settefonti viaduct is an Italian prestressed concrete bridge part of the A1 highway between Bologna and Florence; it was opened to traffic in 1960. It consists of two structurally independent decks with a total length of about 300 m. The main spans are 85 m long and comprise two symmetrical prestressed concrete box girders supporting 50 m long prestressed concrete suspended girders. The box girders are 13.5 m long, between 3.6 m and 5.1 m high, and prestressed by 22 cables. Each cable consists of 12 wires with a diameter of 7 mm, and the average design tension is about 1050 MPa. The suspended span beams are 3.0 m high and prestressed by 9 cables with the same characteristics as above.

As mentioned above, we apply the defined framework to the Limit State of crack opening in prestressed concrete beams (PCBs). The choice of this Limit State is due to these reasons: crack opening in PCBs is frequently monitored, and it is a clear sign of abnormal structural behavior; the Limit State is simple, so it allows to focus on the peculiarities and purposes of the methodology, rather than on the mathematical complexity of the formulation.

Let us consider a simple monitoring system designed to measure the opening variation $\Delta\ell$ of a pre-existing concrete crack (the Observation): a displacement transducer (e.g., a LVDT) connected to the concrete on the two sides of the crack as in Figure 3.

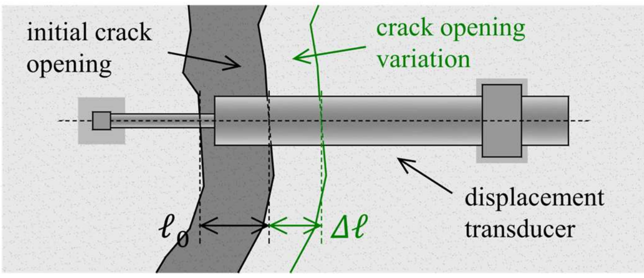

Figure 3. Scheme of the monitoring system for concrete crack opening considered in the application.

To estimate the overall crack opening width w. In this case, the overall crack opening width w is both the State Parameter and the Limit State Parameter ($x = w$). To estimate w, we must know how much the crack has already opened, i.e., the initial crack opening ℓ_0. First, we explain the mathematical models which govern this Limit State, starting from the Interpretative Model $\mathcal{M}$, as reported in Equation (11):

$$\Delta\ell = \mathcal{M}(w) = \mu_{\Delta\ell} + (w - \mu_w) + e_{\Delta\ell} \tag{11}$$

where $\mu_{\Delta\ell}$ is the difference between μ_w and ℓ_0 and $e_{\Delta\ell}$ is the random error of the Interpretative Model. The error $e_{\Delta\ell}$ is defined as a combination of the measurement error e_m, and the model error e_{ℓ_0}, which depends on the uncertainty of the initial crack opening. Since the two model errors add up to each other, the variance of the Interpretive Model $\sigma_{\Delta\ell}^2$ sums σ_m^2 and $\sigma_{\ell_0}^2$, which are the variance of the measurement and the initial crack opening, respectively.

Since the Limit State Parameter x coincides with the State Parameter w, we define the Limit State Function $\mathcal{Z}$ as in Equation (12), where w_{lim} is the maximum acceptable value of crack opening in the PCB. Note that this function follows the scheme in Equation (7), where b_0 is equal to w_{lim} and b is equal to -1.

$$z = w_{lim} - w \tag{12}$$

Now, we can apply the formulation discussed in the previous Section to directly evaluate the Reliability Index β as a function of the Observations $\Delta\ell$. The application of Equation (10) provides the offset β_0 and the Reliability Sensitivity Index $\nabla\beta$ of the Reliability Index β.

$$\beta_0 = \sqrt{1+\lambda} \cdot \frac{w_{lim} - \mu_w}{\sigma_w} \; ; \qquad \nabla\beta = -\sqrt{\frac{\lambda}{1+\lambda}} \cdot \frac{1}{\sigma_{\Delta\ell}} \; ; \tag{13}$$

Note that β_0 and $\nabla\beta$ are a function of the a-dimensional parameter $\lambda = \sigma_w^2/\sigma_{\Delta\ell}^2$, which represents the ratio between the prior variance σ_w^2 and the variance $\sigma_{\Delta\ell}^2$.

Finally, the application of Equation (9) provides the relation between the Reliability Index β and the Observation $\Delta\ell$, as in Equation (14).

$$\beta = \beta_0 + \nabla\beta\,(\Delta\ell - \mu_{\Delta\ell}) = \frac{w_{lim} - \mu_w}{\sigma_w} - \sqrt{\frac{\lambda}{1+\lambda}} \cdot \frac{\Delta\ell + \ell_0 - \mu_w}{\sigma_{\Delta\ell}} \tag{14}$$

Figure 4 shows the function of the Reliability Index β of the Observation $\Delta\ell$ considering, for instance, $w_{lim} = 100\ \mu m$, $\mu_w = 40\ \mu m$, $\sigma_w = 50\ \mu m$, $\ell_0 = 40\ \mu m$, $\sigma_{\ell_0} = 25\ \mu m$, and $\sigma_m = 10\ \mu m$ in Equation (14). Note that when the Interpretative Model uncertainty decreases, the Reliability Sensitivity Index $\nabla\beta$ increase (in modulus). Hence, besides being used in the reliability assessment, $\nabla\beta$ can also be used as an effectiveness index of a monitoring system. It quantitatively expresses the effectiveness of a SHM system in providing helpful information for the reliability assessment of the monitored structure. In other words, an index of the SHM system's potential to provide Observations y that induces a variation in the Reliability Index β. As a result, when infrastructure managers must decide which monitoring system to install among a set of different solutions, they can choose the one with the maximum value of $\nabla\beta$ (in modulus). Let us take a further step forward. We can write the square of Reliability Index Sensitivity as in Equation (15):

$$\nabla\beta^2 = \frac{\lambda}{1+\lambda} \cdot \sigma_{\Delta\ell}^{-2} = g(\lambda) \cdot \sigma_{\Delta\ell}^{-2} \tag{15}$$

where $\sigma_{\Delta\ell}^{-2}$ is the inverse variance (i.e., the accuracy) of the monitoring system, while $g(\lambda)$ is a function of the a-dimensional parameter λ and is defined in Equation (16).

$$g(\lambda) = \nabla\beta^2 \cdot \sigma_{\Delta\ell}^2 = \frac{\lambda}{1+\lambda} \tag{16}$$

Note that when $\lambda \to 0$ (i.e., problem only governed by the prior knowledge), $g(\lambda) \to 0$; consequently, the measurements cannot influence β because $\nabla\beta \to 0$. On the other hand, when $\lambda \to \infty$ (i.e., problem not influenced by the prior knowledge at all), $g(\lambda) \to 1$; consequently, β is only influenced by the Observations.

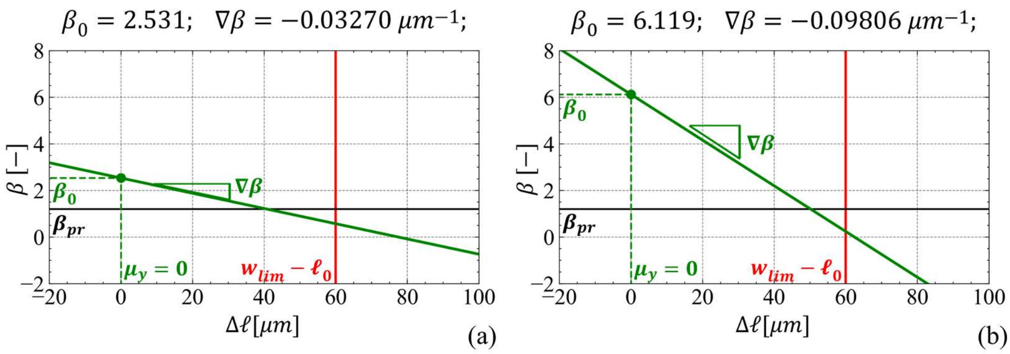

Figure 4. Reliability Index β function of Observation $\Delta\ell$.
(a) with model uncertainty; (b) without model uncertainty ($\sigma_{\ell_0} \to 0\ \mu m$).

CONCLUSIONS

Managers need general and scalable frameworks to be implemented on a large scale, which promptly provide the reliability of their infrastructures based on SHM data. The systematic use of existing numerical methods appears inefficient to fulfil this goal due to the enormous computational efforts they require.

This paper formalizes a GBN-based logical framework for structural reliability analysis based on SHM observations. The assumptions conditioning the framework allow the reliability index to be written directly as a function of the SHM Observations through linear models and in closed form. This function allows for near real-time first-level evaluation of the structural reliability of infrastructures. In addition, the Reliability Index Sensitivity – a parameter of this function – can be seen as a general, novel index for quantifying the effectiveness of an SHM system in providing information on structural reliability with respect to specific Limit States during the monitoring system design. The assumptions of this approach introduce limitations, which will be tested in future studies to have insight into their impact on structural reliability estimation.

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