Approximate Analytical Solution for a Simply Supported Beam on Tensionless Winkler Foundation Under Multiple Concentrated Loads

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ABSTRACT

This work presents an approximate analytical solution for simply supported beams resting on a tensionless Winkler foundation and subjected to multiple point loads. The foundation is modeled as a series of discrete linear independent springs and gaps may exist between the beam and the springs. The two-stage technique approximates the response of the tensionless foundation: in the first stage, the technique identifies the contact and noncontact regions based on an iterative procedure, while in the second stage, the technique redistributes the springs in the contact region only. Results are validated and compared with a benchmark solution. The numerical examples confirm the efficiency and accuracy of the proposed method.

INTRODUCTION

The Winkler foundation model, which comprises a continuous array of independent linear springs, is widely used in different engineering applications, such as railroads, pipelines, and piles [1–4]. The Winkler model assumes that the foundation transmits both compressive and tensile forces. However, in some engineering applications, such as ballasted railroads, gaps may exist and the assumption of tensile reaction becomes unrealistic [5,6]. In this context, the tensionless Winkler foundation is more realistic, since it reacts in compression only. However, restricting reactions to compressive response introduces nonlinearity and discontinuity to the beam-foundation interaction [7–10].

Several studies have been developed for beams on tensionless elastic foundations. For an infinite beam, Tasi and Westmann (1967) presented a solution for an infinite beam resting on a tensionless Winkler foundation and subjected to a concentrated load [7]. Weitsman (1970) presented an analytical solution for an infinite beam on a tensionless foundation subjected to concentrated and uniformly distributed loads [8]. In

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recent work, Ma et al. (2009) investigated the static response of an infinite beam resting on a tensionless foundation and carrying arbitrary complex loads [11]. For finite beams, Zhang and Murphy (2004, 2013) presented an analytical solution for a static response of a finite beam on a tensionless Winkler foundation under concentrated load, with a zero/nonzero gap at the interface [12,13]. Attar et al. proposed a method for static and dynamic analysis of a beam-like on a tensionless viscoelastic foundation, with either zero or nonzero gap at the interface [14]. Many researchers have developed solutions for infinite and finite beams on tensionless foundations [6,15–17].

This paper presents an approximate analytical solution for a simply supported beam resting on a tensionless Winkler foundation and subjected to multiple concentrated loads. The foundation is modeled as discrete linear independent springs, which can be used to estimate the static response of a tensionless Winkler foundation. Utilizing a two-stage technique, the response of the tensionless Winkler foundation with either zero or nonzero gap can be approximated. Results are validated with a benchmark solution in the literature, demonstrating the method's reliability and accuracy.

THEORETICAL DERIVATIONS

Consider a simply supported beam resting on a series of discrete elastic linear springs, with spring stiffness k_{sn} and gap y_0, and subjected to a concentrated load P_j at distance x_{Pj} from the left support, as shown in Fig. 1. The beam has length L and flexural rigidity EI.

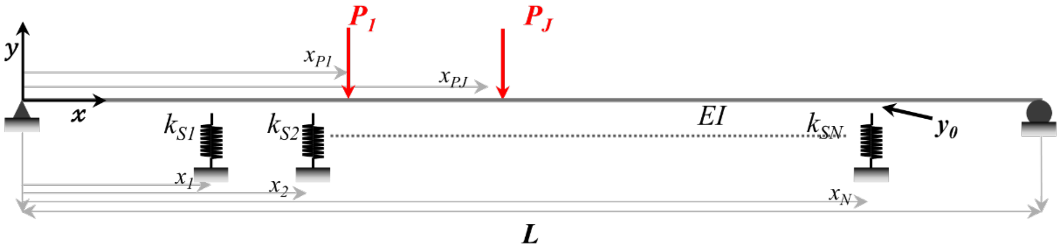

Figure 1. Simply supported beam resting on discrete springs and subjected to a concentrated load.

The internal bending moment due to a concentrated load, P, is obtained by Eq. (1):

$$M(x) = \begin{cases} M_1(x) = \frac{Px(L-x_P)}{L}, & 0 \le x \le x_P \\ M_2(x) = \frac{Px_P(L-x)}{L}, & x_P \le x \le L \end{cases} \tag{1}$$

Eq. (1) can be written as follows:

$$M(x) = M_1(x) + P(x_P - x)H(x - x_P) \tag{2}$$

Where H(.) is the Heaviside function. By utilizing the moment-curvature relationship along with boundary conditions, as shown in Eqs. (3 and 4):

$$M(x) = EI\frac{d^2y(x)}{dx^2} \tag{3}$$

$$y_P(0) = 0 \qquad\qquad y_P(L) = 0 \tag{4a, b}$$

The beam deflection due to the concentrated load is derived as shown in Eq. (5):

$$y_P(x) = \frac{P}{6EI}\left[\left(1 - \frac{x_P}{L}\right)x^3 - (x_P - x)^3 H(x - x_P)\right] + C_1 x + C_2 \tag{5}$$

Where:

$$C_1 = \frac{P}{6EIL}[(L^3 - 3L^2 x_P + 3Lx_P^2 - x_P^3)H(L - x_P) + x_P^3 H(-x_P) + L^2 x_P - L^3] \tag{6}$$

$$C_2 = \frac{P x_P^3 H(-x_P)}{6EI} \tag{7}$$

The beam deflection due to spring support can be derived by substituting Eq. (8) into Eqs. (5, 6, and 7)

$$P = F_{Sn} = k_{Sn}\delta_{xn} \qquad\qquad x_P = x_n \tag{8a, b}$$

Where F_{Sn}, k_{Sn} *and* δ_{xn} are the spring restoring force, the spring constant, and the spring displacement, respectively. The index *n = 1, 2, ..., N,* and *N* is the total number of springs. The beam deflection due to spring support is obtained:

$$y_{Sn}(x) = \frac{F_{Sn}}{6EI}\left[\left(1 - \frac{x_n}{L}\right)x^3 - (x_n - x)^3 H(x - x_n)\right] + C_1 x + C_2 \tag{9}$$

Where:

$$C_1 = \frac{F_{Sn}}{6EIL}[(L^3 - 3L^2 x_n + 3Lx_n^2 - x_n^3)H(L - x_n) + x_n^3 H(-x_n) + L^2 x_n - L^3] \tag{10}$$

$$C_2 = \frac{F_{Sn} x_n^3 H(-x_n)}{6EI} \tag{11}$$

Utilizing the superposition principle, the displacement at each spring can be solved using Eq. (12):

$$\begin{gathered} \delta_{xn} + y_0 = \textstyle\sum_{j=1}^{J} y_{Pj}(x_n) + \sum_{n=1}^{N} y_{Sn}(x_n) \\ \vdots \\ \delta_{xN} + y_0 = \textstyle\sum_{j=1}^{J} y_{Pj}(x_N) + \sum_{n=1}^{N} y_{Sn}(x_N) \end{gathered} \tag{12}$$

Where *y0* is the initial gap between the beam and the foundation. The values are then substituted into Eq. (13).

$$y(x) = \textstyle\sum_{j=1}^{J} y_{Pj}(x) + \sum_{n=1}^{N} y_{Sn}(x) \tag{13}$$

Where $j = 1,2, \ldots, J$ and J is the total number of concentrated loads and $y(x)$ is the total deflection for the system shown in Fig. (1). To approximate a tensionless Winkler foundation, a two-stage technique is proposed. In the first stage, the technique determines the lift-off points based on an iterative technique, and the spacing between the springs is calculated as follows:

$$l = \frac{L}{N} \tag{14}$$

By using the discrete springs, the subgrade modulus can be approximated as follows:

$$k_{sn} = Kl \tag{15}$$

All springs are assumed to be connected to the beam, and the technique iteratively removes any spring that is in tension and then repeats the. The technique continuously updates until all the springs are in compression. For each iteration, the lift-off points can be updated and solved numerically by equating Eq. (13) to the initial gap as follows:

$$y(x) = y_0 \tag{16}$$

In the second stage, the springs were redistributed within the contact region only. The technique iteratively increases the springs in the contact region and recalculates the spacing between the springs. The technique continuously updates the system until a maximum relative change in the lift-off points satisfies the condition:

$$\varepsilon = max\left(\frac{|LF_i - LF_{i-1}|}{|LF_{i-1}|}\right) \leq T \tag{17}$$

Where LF_i and LF_{i-1} are the lift-off point values at iterations i and i-1, respectively and T is the convergence tolerance. If the condition is met, no iterations are required, as the solution achieves optimal accuracy. A visual illustration of the proposed method is shown in Fig (3), corresponding to the example in Fig (2a)

NUMERICAL EXAMPLE

Three representative examples illustrate the technique. The results of the first and second examples are compared with the benchmark solution that has been published by Zhang and Murphy (2004) while in the third example, the results are compared with FE. In their work of Zhang and Murphy (2004), the coordinate origin is considered at x_P, therefore, the origin of the present formulation is shifted to x_P. The results are normalized with respect to the foundation stiffness parameter $\beta = \sqrt[4]{\frac{K}{4EI}}$. The nondimensional displacement, $\bar{y}(x)$, is plotted against the nondimensional position, $\boldsymbol{\xi}$, as shown in Fig. (2a, b). The nondimensional beam length $\bar{L} = 30$ and the load is located at midspan in both cases. In Fig. (2a), the nondimensional force, F, equals 0.01, there is zero initial gap, and T is 1e-5. As shown in Fig (2a), the proposed method has an excellent agreement with an analytical solution which confirms the accuracy and capability of the proposed method. Only 16 springs were utilized to achieve the optimal solution. In Fig (2b), a nonzero gap, $\bar{y}_0$ = - 0.05, and nondimensional Force, F = 0.1, are considered. As shown in Fig. (2b), the obtained results aligned well with analytical solutions. However, the analytical solution is only applicable for a single concentrated load and one contact region. For further validation of multiple concentrated loads, a final example illustrates the proposed method and is compared with an FE method (SAP2000).

Consider a pinned-pinned beam resting on a tensionless Winkler foundation and subjected to two points loads, P_1 = 30000 lb and P_2 = 10000 lb, at $x_{P1} = \frac{L}{4}$ and $x_{P2} = \frac{3L}{4}$ from the left support, respectively. The beam has a length L = 60 ft, a young's modulus $E = 30 \times 10^6\, psi$ and second moment of Inertia $I = 95\, in^4$. The foundation stiffness is $K = 5000\, psi$. The deflection is plotted against the normalized position, $\frac{x}{L}$, as shown in Fig. (3). As illustrated in Fig. (3), the proposed method has an excellent match with the FE results using 50 springs distributed within the contact regions and a convergence tolerance of 10^{-5}, confirming its accuracy and efficiency. Overall, the results demonstrate the efficiency and accuracy of the proposed method. Higher accuracy can be achieved with a lower Tolerance value.

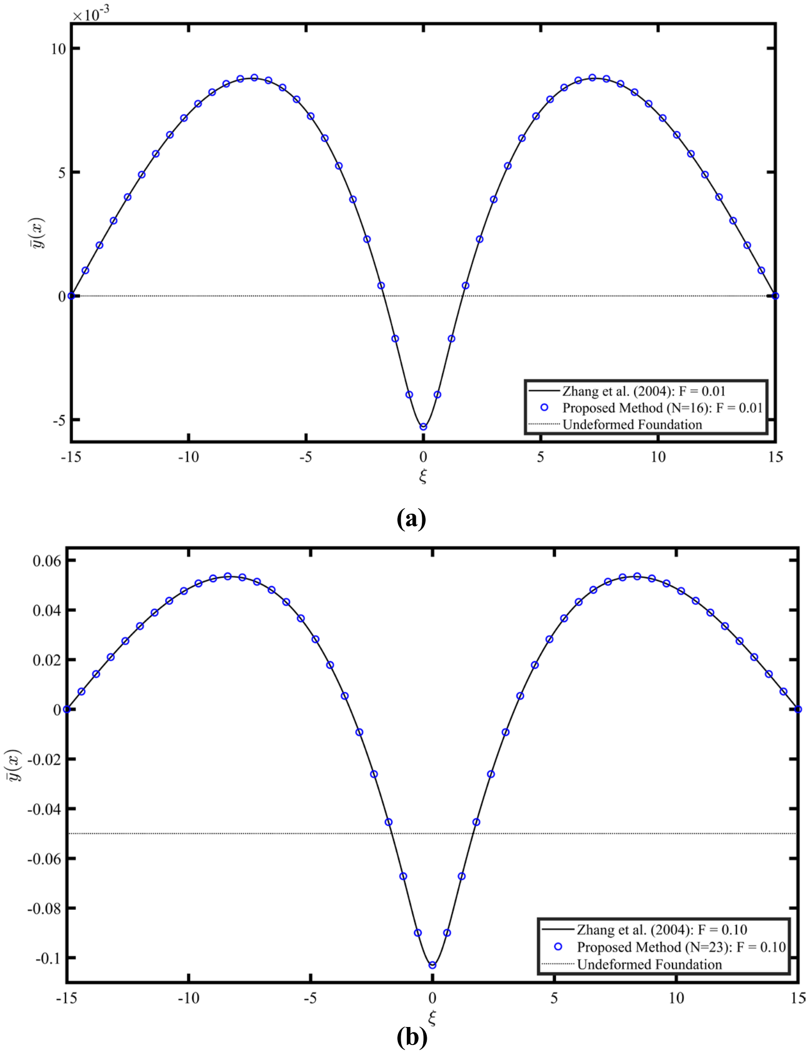

Figure 2. (a) Beam deflection under a concentrated, F = 0.01, load at midspan, with $\bar{y}_0 = 0$. (b) Beam deflection under a concentrated, F = 0.1, load at midspan, with $\bar{y}_0 = -0.05$

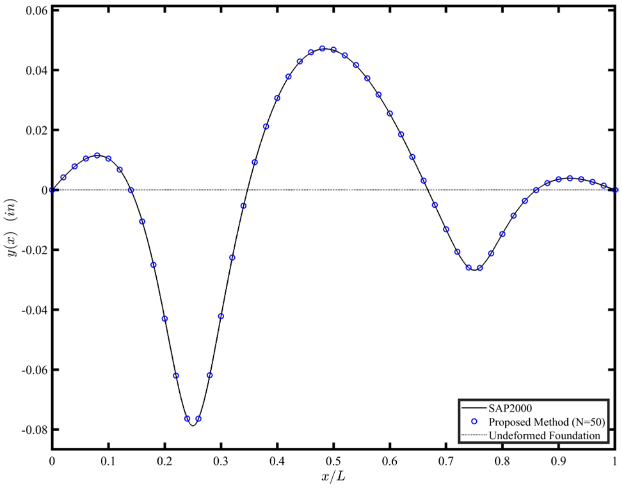

Figure 3. Beam deflection of a pinned-pinned beam resting on a tensionless Winkler foundation and subjected to two concentrated loads

VISUAL ILLUSTRATION OF METHOD

In this section, a visual explanation of the proposed technique on the example in the previous section in Fig (2a) is considered. In stage 1, the starting number of springs was 30. The technique is iteratively updating the system and removing any spring that experiences a tension force, as shown in Fig (4). The lift-off points were determined for each iteration. When all springs are in compression, the technique redistributed the springs, stage 2, in the contact region starting with 3 springs and with an increment of 1 spring for each iteration until the condition in Eq. (15) is satisfied, as shown in Fig (5). In stage 2, only the first and last iterations are shown for brevity purposes.

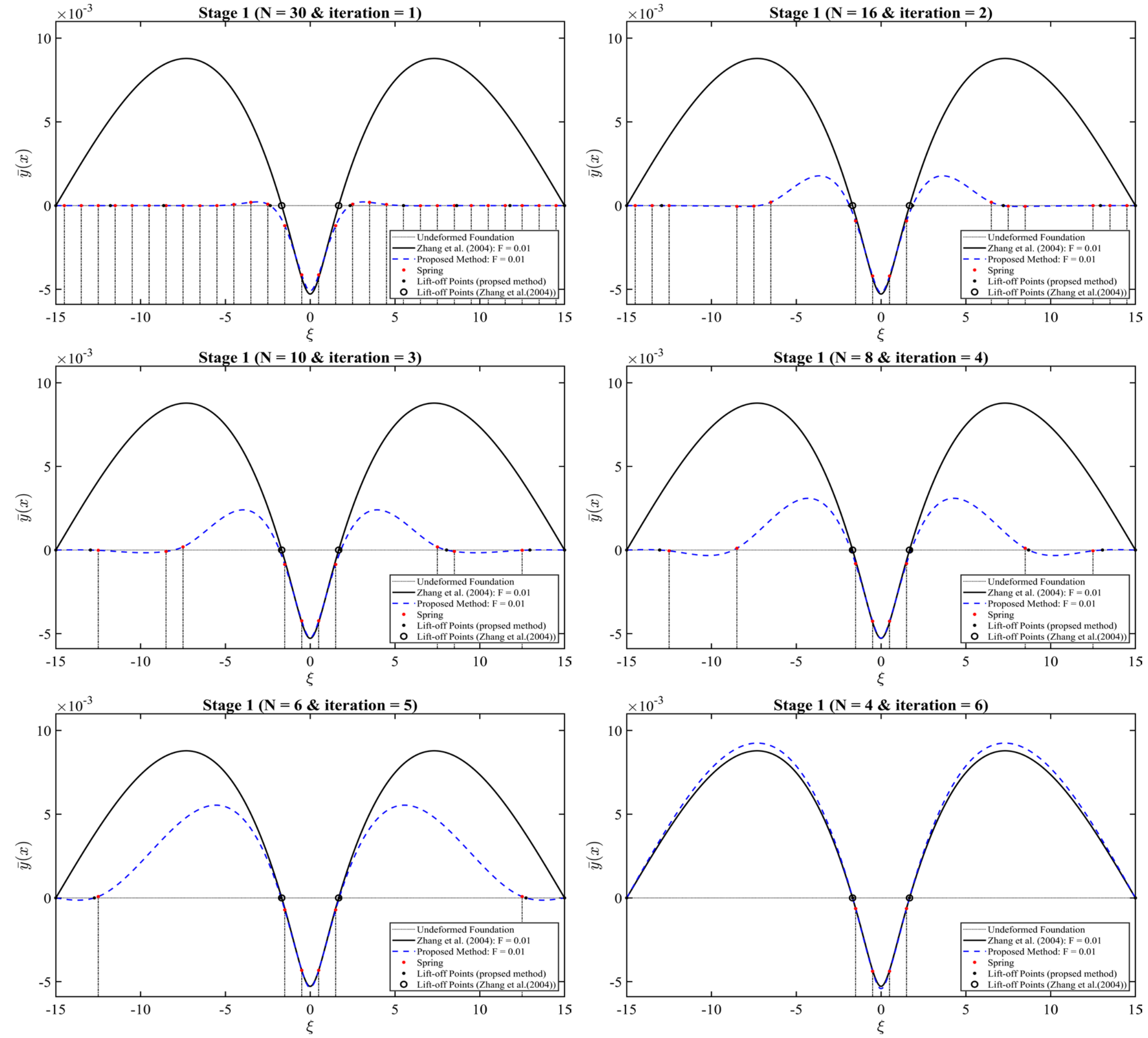

Figure 4. Visual illustration for stage 1

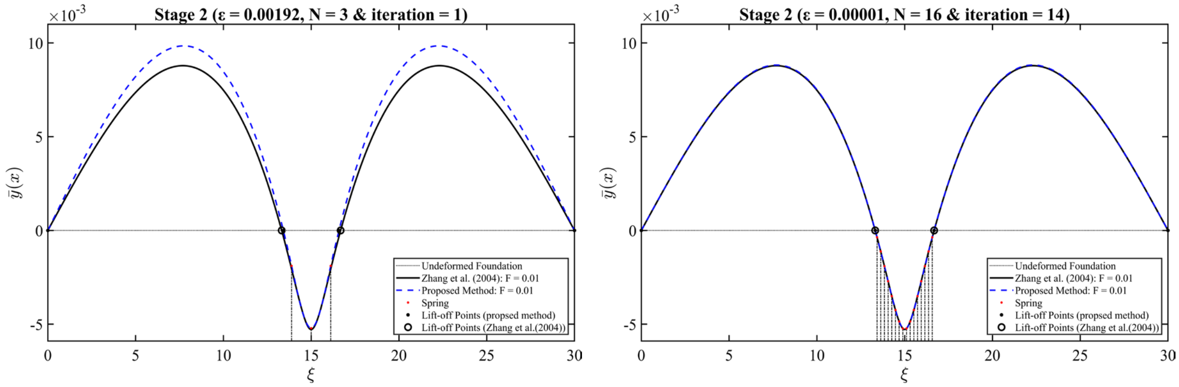

Figure 5. Visual illustration for stage 2

CONCLUSION

In this work, an approximate analytical solution has been developed for a static deflection of a simply supported beam on a tensionless Winkler foundation with either zero or nonzero gap at the interface. The foundation is modeled as a discrete linear independent spring. Two-stage technique were proposed. In the first stage, the technique identifies the contact and noncontact regions. In the second stage, the technique redistributes the springs in the contact region and iteratively increases the springs, and an optimal solution is achieved based on a predefined Tolerance. The numerical results confirm its accuracy and validity.

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